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Configurational studies of the Potts models

C Domb

Wheatstone Physics Laboratory, King's College, Strand, London WC2R 2LS, UK

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Abstract. Eigenvalues and eigenvectors are derived for the q -state two-level model introduced by Potts in 1952, and it is shown that for any net the partition function depends only on the topology and not on any two-degree vertices. This enables a simple method to be used calculating the partition functions of standard star topologies. The second q -orientation model introduced by Potts (termed the *planar Potts model*) is discussed, and it is shown that the same property holds. Partition functions for certain star topologies are derived for this model. By considering the asymptotic form of the coefficients in high-temperature expansions of the partition function, estimates are obtained of the critical temperatures of these models in terms of the geometrical properties of self-avoiding walks.

1. Introduction

In 1951 when the present author was at Oxford he pointed out to his research student R B Potts that the transformation discovered by Kramers and Wannier (1941) for the two-dimensional Ising model could be generalized to a planar vector model having three symmetric orientations at angles of 0 , $2\pi/3$, $4\pi/3$ with the axis. Hence the Curie temperature could be located for this model. He suggested that it might be possible to extend the result to a planar vector model with q symmetric orientations.

After a detailed investigation Potts (1952) came to the conclusion that the transformation did not generalize to a planar vector model with q orientations, but instead to a q -state model in which there are two different energies of interaction which correspond to nearest neighbours being in the same state or different states; the case $q = 4$ for this model had been considered previously by Ashkin and Teller (1943). For the *planar* model with $q = 4$ it was possible to locate the Curie temperature by an alternative method, but this failed for higher values of q .

Once the transformation had been discovered and the Curie point located it seemed reasonable to expect that a complete solution would be forthcoming following the previous history of the Ising model. Surprisingly, even though two decades have now elapsed only one exact solution has emerged for the planar model with $q = 4$. The partition function for this model reduces to the square of a standard Ising partition function (Betts 1964). A useful insight into the model comes from the observation by Kasteleyn at the Aachen Conference on Statistical Mechanics, June 1964, that it is isomorphic with two uncoupled Ising spin systems; the result is therefore valid for any network.

For many years the above models attracted little attention because they did not seem to represent an interaction of physical significance. In 1971 Mittag and Stephen undertook

an algebraic investigation of the duality transformation, and provided a topological description analogous to that of Onsager for the Ising model (Wannier 1945). In order to differentiate between the two types of model they referred to the q -state two-energy model as the *Potts model*, and the planar q -orientation model as the *Potts vector model*. We are unhappy with this terminology since the first model is also a vector model. The case of four states corresponds to four symmetric tetrahedral directions in three dimensions, of five states to five symmetric directions of a simplex in four dimensions, . . . , of q states to q symmetric directions of a simplex in $(q - 1)$ dimensions. Therefore we prefer to call the q -orientation model the *planar Potts model*; we retain the terminology *Potts model* of Mittag and Stephen for the two-energy model, but if we need to be more specific we will call it the *standard Potts model*.

Very recently a number of additional papers have appeared dealing with the Potts model. Straley and Fisher (1973) derived low-temperature series expansions for the three-state case on the simple quadratic lattice using different coupling fields for each orientation. They used these series to explore critical behaviour which they contrasted with the predictions of Landau theory. Golner (1973) endeavoured to use the Wilson renormalization group technique to explore the critical behaviour of the model. Baxter (1973) related the model to ferroelectric models which have been solved exactly, and hence obtained strong indications that the Potts model has a first-order transition when $q > 4$. Alexander and Yuval (1973) have given a physical interpretation of the three-state case and suggested that it could be usefully related to certain liquid crystal transformations; they also derived series expansions and considered the model with antiferromagnetic interactions. Finally Enting (1974) has used the code method (Sykes *et al* 1965, 1973, Domb 1974, chap 1) to derive high-field expansions and estimate the exponent δ .

Both the standard and planar models have a simplifying feature possessed by the n -dimensional classical vector model that only star lattice constants need be taken into account in computing high-temperature series expansions for the partition function (see Domb 1974, chap 1). The weights of these lattice constants can be determined using methods similar to those which have been developed for the classical vector model (Domb 1972b) and a series can be derived for the Curie temperature of a given lattice in terms of the geometrical properties of self-avoiding walks on the lattice. The latter has a special interest since the series expansions which have been derived for the three-state model converge only slowly.

It is the purpose of the present paper to provide this configurational background for both Potts models, and hence to pave the way for the calculation of high-temperature expansions for the standard two- and three-dimensional lattices thus exploiting the extensive available tables of star lattice constants.

2. Potts model: diagonalization of interaction matrix

Although the eigenvalues and eigenvectors of the interaction matrix were determined by Potts (1952), we shall re-derive them rapidly so as to draw attention to some features which have not been pointed out before. For the simplex of q vectors in $(q - 1)$ dimensions there is a definite relation between $-J_0$ the energy of interaction of spins in the same state, and $-J_1$ the energy of interaction in different states; for example, with $q = 3$, $J_1 = -J_0/2$. However, the general theory can be developed for arbitrary J_0, J_1 . The $q \times q$ interaction matrix \mathbf{V} has diagonal elements $x_0 = \exp \beta J_0$ and non-diagonal elements $x_1 = \exp \beta J_1$ ($\beta = 1/kT$). It has the symmetry of the permutation group, and the

eigenvectors are

$$\left. \begin{aligned} \xi_1 &= (1, 1, \dots, 1) \\ \xi_2 &= (1, \omega, \omega^2, \dots, \omega^{q-1}) \\ &\vdots \\ \xi_t &= (1, \omega^t, \omega^{2t}, \dots, \omega^{t(q-1)}) \\ &\vdots \end{aligned} \right\} \omega = \exp(2\pi i/q). \tag{1}$$

It is important to note that these eigenvectors are independent of J_0, J_1 .

The eigenvalues have two nonzero values,

$$\begin{aligned} \lambda_0 &= x_0 + (q-1)x_1, \\ \lambda_1 &= (x_0 - x_1)(q-1 \text{ times}). \end{aligned} \tag{2}$$

The matrix \mathbf{T} which diagonalizes \mathbf{V} has the form

$$\mathbf{T} = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{q-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega^t & \omega^{2t} & \dots & \omega^{t(q-1)} \end{pmatrix} \tag{3}$$

and its inverse is equal to its complex conjugate

$$\mathbf{T}^{-1} = \bar{\mathbf{T}}. \tag{4}$$

Hence we can write

$$\mathbf{V} = \mathbf{T}\mathbf{\Lambda}\bar{\mathbf{T}}, \tag{5}$$

where $\mathbf{\Lambda}$ is the diagonal matrix

$$\begin{pmatrix} \lambda_0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix}. \tag{6}$$

From this it follows that

$$\mathbf{V}^r = \mathbf{T}\mathbf{\Lambda}^r\bar{\mathbf{T}} \tag{7}$$

and this has elements

$$\begin{aligned} (\mathbf{V}^r)_{ii} &= \frac{1}{q}[\lambda_0^r + (q-1)\lambda_1^r] \\ (\mathbf{V}^r)_{ij} &= \frac{1}{q}(\lambda_0^r - \lambda_1^r) \quad (i \neq j). \end{aligned} \tag{8}$$

We now consider the partition functions of simple finite nets. For a simple polygon of n identical systems

$$\begin{aligned}
 Z(n)_p &= \sum_{i,j,\dots,k} V_{ij}V_{jk}\dots V_{ki} = \text{Tr}(\mathbf{V}^n) \\
 &= \lambda_0^n + (q-1)\lambda_1^n = \lambda_0^n[1 + (q-1)w^n]
 \end{aligned}
 \tag{9}$$

where

$$w = \frac{\lambda_1}{\lambda_0} = \frac{x_0 - x_1}{x_0 + (q-1)x_1}
 \tag{10}$$

If the interactions among a group of r of the rings are changed from J_0, J_1 to J'_0, J'_1 the partition function is changed to

$$\begin{aligned}
 Z(n)_p &= \text{Tr}(\mathbf{V}^r\mathbf{V}^s) = \text{Tr}(\Lambda^r\Lambda^s) \\
 &= \lambda_0^r\lambda_0^s + (q-1)\lambda_1^r\lambda_1^s = \lambda_0^r\lambda_0^s[1 + (q-1)w^r w^s] \quad (r+s = n).
 \end{aligned}
 \tag{11}$$

Let us now consider a set of n links forming a θ topology (figure 1, Sykes *et al* 1966). The partition function is given by

$$Z(r, s, t)_\theta = \sum_{i,j} (\mathbf{V}^r)_{ij}(\mathbf{V}^s)_{ij}(\mathbf{V}^t)_{ij}
 \tag{12}$$

where the summation is over all q^2 pairs of values of i, j . From (8) we find that

$$\begin{aligned}
 Z(r, s, t)_\theta &= \frac{1}{q^2} \{ [\lambda_0^r + (q-1)\lambda_1^r][\lambda_0^s + (q-1)\lambda_1^s][\lambda_0^t + (q-1)\lambda_1^t] \\
 &\quad + (q-1)(\lambda_0^r - \lambda_1^r)(\lambda_0^s - \lambda_1^s)(\lambda_0^t - \lambda_1^t) \}.
 \end{aligned}
 \tag{13}$$

We readily see that if the interactions differ on the three chains of the θ topology the partition function is obtained by writing $\lambda_0(J), \lambda_1(J)$ in the first bracket, $\lambda_0(J'), \lambda_1(J')$ in the second, and $\lambda_0(J''), \lambda_1(J'')$ in the third. Likewise, if the interaction is changed in a portion of one chain, say s' and s'' , we obtain the new partition function by replacing λ_0^s and λ_1^s by $\lambda_0^{s'}\lambda_0^{s''}$ and $\lambda_1^{s'}\lambda_1^{s''}$ respectively. Also if we ignore all vertices of degree two, and calculate the partition function for a net with interactions J, J', J'' we can immediately write down the partition function for the general net with r, s, t links by replacing $\lambda_0(J), \lambda_1(J)$ by λ_0^r, λ_1^r , $\lambda_0(J'), \lambda_1(J')$ by λ_0^s, λ_1^s and $\lambda_0(J''), \lambda_1(J'')$ by λ_0^t, λ_1^t (figure 1). For such a net it is convenient to write (13) in the form

$$Z(r, s, t)_\theta = \frac{1}{q} \lambda_0(J)\lambda_0(J')\lambda_0(J'')[1 + (q-1)(ww' + ww'' + w'w'') + (q-1)(q-2)ww'w'']
 \tag{14}$$

where w is defined in (10).

We now consider the partition function for any net

$$Z(G) = \sum_{i,j,\dots,k} V_{ij}V_{jk}\dots V_{ki},
 \tag{15}$$

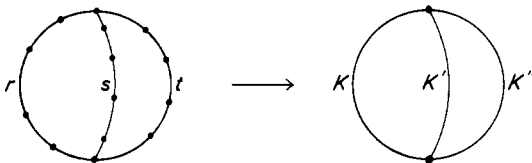


Figure 1. Replacement of two degree vertices by different interactions for a θ topology.

the sum being taken over all links of the net ; it is convenient to write, for any interaction J ,

$$V_{ij}(J) = \lambda_0(J)[\epsilon_{ij} + w(J)\eta_{ij}] \tag{16}$$

where $\epsilon_{ij} = 1$ for all i, j and $\eta_{ij} = -1, (i \neq j), \eta_{ii} = (q - 1)$. We note that ϵ_{ij} and η_{ij} are independent of the interaction, and

$$\sum_j \eta_{ij} = 0 \tag{17}$$

for all i . We can use (16) to expand the partition function in terms of sub-graphs of the net in the same way as for the Ising model (Domb 1974, chap 6). Relation (17) ensures that all graphs with vertices of degree unity give zero contribution. However, higher-order odd vertices do not give zero contribution when $q > 2$, as can be verified from (14).

The properties established above for polygons and θ topologies will now be seen to hold more generally. If any interaction J is changed to J' the new partition function is derived by changing the appropriate w into w' and the multiplying factor $\lambda_0(J)$ into $\lambda_0(J')$; and the form of the partition function is determined essentially by the topology of the net, the insertion of two-degree vertices giving rise to a trivial change.

We mention one additional property which follows immediately from the symmetry of the interactions. For any articulated graph $G \circ G'$ constructed from G, G' by identifying a common vertex, the partition function is given by

$$Z(G \circ G') = \frac{1}{q} Z(G)Z(G') \tag{18}$$

This ensures that in the expansion of $\ln Z(G)$ for any net G only star sub-graphs need be taken into account (Domb 1974, chap 1).

We shall now show that the general properties established in this section enable the partition functions for star topologies to be derived successively and simply.

3. Partition functions of star topologies

From the previous section we see that the partition function for a net G can be put in the form

$$Z(G) = \left(\prod_r \lambda_0(J^{(r)}) \right) F(G|w^{(r)}) \tag{19}$$

Here we are confining attention to a *star topology* G which has no vertices of degree 2, and all of whose bonds have different interactions $J^{(r)}$; the product is taken over all these bonds. The function $F(G|w^{(r)})$ characteristic of the topology G consists of a sum of partial products of the $w^{(r)}$ with appropriate coefficients,

$$F(G|w^{(r)}) = 1 + \sum a_{rs} w^{(r)} w^{(s)} + \sum a_{rst} w^{(r)} w^{(s)} w^{(t)} + \dots \tag{20}$$

Only terms corresponding to closed graphs enter into (20).

We build up the star topologies according to cyclomatic number c (Sykes *et al* 1966). If we put one of the $w^{(r)}$ equal to zero in (20), this corresponds to removing a bond from G which leaves a graph of cyclomatic number $c - 1$. Hence by putting each of the $w^{(r)}$ successively equal to zero, all the coefficients for a star of cyclomatic number c can be

determined from stars of cyclomatic number $(c - 1)$ with the exception of the highest coefficient, the product of all the $w^{(r)}$. From (16) this coefficient is given by

$$\sum_{i,j} \prod_r \eta_{ij}^{(r)} \tag{21}$$

where the product r is over all bonds of the net G and each i and j takes q values.

It is straightforward but tedious to evaluate (21) directly for a given net. We can write

$$\eta_{ij} = -\frac{1}{q} \epsilon_{ij} + \delta_{ij} \tag{22}$$

and proceed to separate out terms in the product (21); these correspond to dropping 1, 2, 3 ... lines from G and seeing how many connected clusters remain. The procedure closely parallels the calculation of the mean number of connected clusters in percolation theory (Essam 1972).

However, it is simpler to use the method introduced by Domb (1972b) for the calculation of partition functions for the classical vector models. We use the term *laddering* to denote the replacement of a single interaction J^* by a pair of interactions J' and J'' in parallel (figure 2). Thus x_0^* is replaced by $x_0'x_0''$ and x_1^* by $x_1'x_1''$. It is easy to deduce from (10) the transformation

$$w^* = \frac{w' + w'' + (q - 2)w'w''}{1 + (q - 1)w'w''} \tag{23}$$

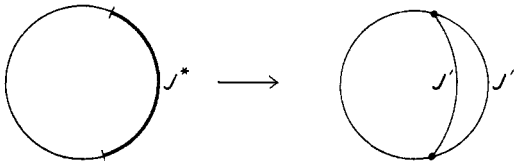


Figure 2. Ladder transformation.

Applying this to the simple polygon with interactions J, J^* for which

$$F(p|w^{(r)}) = 1 + (q - 1)ww^*, \tag{24}$$

we readily find that for the θ topology,

$$F(\theta|w^{(r)}) = 1 + (q - 1)(ww' + ww'' + w'w'') + (q - 1)(q - 2)ww'w'', \tag{25}$$

in agreement with the direct calculation (14). If we apply a double laddering procedure to the polygon (figure 3) we obtain from

$$F(p|w^{(r)}) = 1 + (q - 1)w_1^*w_2^*w_3w_3', \tag{26}$$

the characteristic function for the β topology

$$F(\beta|w^{(r)}) = 1 + (q - 1)(w_1w_1' + w_2w_2') + (q - 1)^2w_1w_1'w_2w_2' + (q - 1)[w_1 + w_1' + (q - 2)w_1w_1'] [w_2 + w_2' + (q - 2)w_2w_2'] w_3w_3'.$$

The corresponding function for the γ topology is obtained by making J_3 infinite, ie $w_3 = 1$, and for the δ topology by making J_3 and J_3' infinite, ie $w_3 = w_3' = 1$.

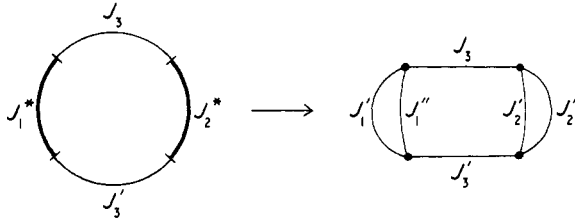


Figure 3. Use of two ladder transformations to derive the β topology.

To deal with non-ladder topologies we make a suitably chosen bond infinite, and obtain a relation for the unknown coefficient in terms of a coefficient of a known or laddered topology and a coefficient of a graph of lower cyclomatic number corresponding to removing the bond. For example, for the largest coefficient a_{123456} of the α topology we make bond 34 infinite, and relate to coefficients in the γ and θ topologies (figure 4)

$$a_{123456} = (q-1)(q-2)^2 - (q-1)(q-2) = (q-1)(q-2)(q-3). \tag{27}$$

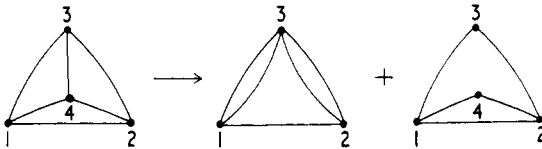


Figure 4. Allowing a bond to become infinite in the α topology.

We now run briefly through the graphs of cyclomatic number 4 (figure 5). All except A, B and F can be obtained by laddering. By making bond 15 of F infinite we relate the highest coefficient to the N and α topologies and deduce that its value is

$$(q-1)(q-2)(q^2 - 5q + 7). \tag{28}$$

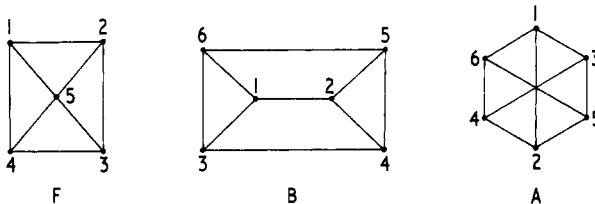


Figure 5. Non-ladder topologies of cyclomatic number 4.

(As a check we can make 12 infinite and relate to the J and γ topologies.) By making bond 12 of the B topology infinite we relate to the F and β topologies, and deduce that the highest coefficient is

$$(q-1)(q-2)(q-3)^2. \tag{29}$$

By making bond 12 of the A topology infinite we relate to the F and α topologies, and deduce that the highest coefficient is

$$(q-1)(q-2)(q^2-6q+10). \tag{30}$$

Having provided a mechanism for writing down the partition functions of star topologies we can use the same method as for the Ising model (Sykes *et al* 1974) to derive extensive high-temperature series expansions for $\ln Z$ for standard two- and three-dimensional lattices.

4. Planar Potts model

For the planar Potts model with q states the interaction between spins in states i and j is

$$J_{ij} = -J \cos(\theta_i - \theta_j) \quad \left(\theta_i = \frac{2\pi i}{q} \right). \tag{31}$$

Hence the interaction matrix \mathbf{V} has elements

$$V_{ij} = \exp \beta J \cos(\theta_i - \theta_j), \tag{32}$$

which depend on $|\theta_i - \theta_j|$ only, and therefore it has the symmetry of the permutation group. The eigenvectors are the same as in (1), and are independent of the interaction. The same would apply to any model whose interaction depends on $|\theta_i - \theta_j|$, for example the model recently considered by Guttman *et al* (1972).

The eigenvalues of the interaction matrix are thus given by

$$\left. \begin{aligned} \lambda_0 &= \sum_{s=0}^{q-1} f(s) \\ \lambda_1 &= \sum_{s=0}^{q-1} f(s)\omega^s \\ &\vdots \\ \lambda_t &= \sum_{s=0}^{q-1} f(s)\omega^{ts} \\ &\vdots \end{aligned} \right\} f(s) \equiv \exp\left(\beta J \cos \frac{2\pi s}{q}\right). \tag{33}$$

Since $f(s)$ is equal to $f(q-s)$, we see that λ_t is equal to λ_{q-t} . Hence $\lambda_1, \lambda_2, \dots$, are doubly degenerate. The general pattern of behaviour depends on whether q is odd or even. In the former case λ_0 is the only non-degenerate eigenvalue, the remainder being doubly degenerate. We can write

$$\begin{aligned} \lambda_0 &= \exp \beta J + 2 \sum_{s=1}^{(q-1)/2} f(s) \\ \lambda_t &= \exp \beta J + 2 \sum_{s=1}^{(q-1)/2} f(s) \cos(2\pi st/q) \quad (t = 1, 2, \dots, (q-1)/2). \end{aligned} \tag{34}$$

In the latter case both λ_0 and $\lambda_{q/2}$ are non-degenerate and the remainder are doubly degenerate. We now have

$$\lambda_0 = \exp \beta J + \exp -\beta J + 2 \sum_{s=1}^{q/2-1} f(s)$$

$$\lambda_t = \exp \beta J + (-1)^t \exp -\beta J + 2 \sum_{s=1}^{q/2-1} f(s) \cos\left(\frac{2\pi st}{q}\right) \quad (t = 1, 2, \dots, \frac{1}{2}q - 1) \quad (35)$$

$$\lambda_{q/2} = \exp \beta J + (-1)^{q/2} \exp -\beta J + 2 \sum_{s=1}^{q/2-1} (-1)^s f(s).$$

The difference in behaviour between odd and even q is to be expected because of the different symmetries in the two cases. When q is even the model is invariant under a reversal of magnetic field, whereas when q is odd it is not. Likewise we can expect special behaviour when q is a multiple of 4 since the model is then invariant under a rotation by 90° of magnetic field.

We have noted that for $q = 3$ this model is identical with the standard Potts model, and for $q = 4$ it is isomorphic with a pair of non-interacting Ising models; for $q = 6$ it can be regarded as a triplet of Ising models with a coupling between them, and so on for higher even values of q . As $q \rightarrow \infty$ we are led to the plane-rotator model first introduced by Vaks and Larkin (1965). From (34) or (35) we then find that the eigenvalues tend to the limiting values

$$\lambda_r = I_r(\beta J) \quad (r = 0, 1, 2, \dots), \quad (36)$$

the lowest state being non-degenerate and the higher states all doubly degenerate. This agrees with the result of Joyce (1967).

The matrix \mathbf{T} given by (3) diagonalizes \mathbf{V} , and the general arguments of § 2 apply equally to the planar Potts model. In particular the partition function for any net is determined by the topology of the net and all two degree vertices can be ignored; and if the interaction of any portion of a net is changed from J to J' , the new partition function is derived by changing any $\lambda_r(J)$ in the old partition function into $\lambda_r(J')$. Because of this property we can use the laddering technique of § 3 to derive the partition functions of most of the star topologies. However, the problem is more complex since we do not now have a single variable w , but a number of variables

$$w_1 = \lambda_1/\lambda_0, \quad w_2 = \lambda_2/\lambda_0, \quad \dots, \quad w_t = \lambda_t/\lambda_0, \quad \dots, \quad (37)$$

we shall therefore discuss the laddering procedure in more detail.

5. Laddering in the planar model

From relation (7) we can calculate the elements of the matrix (\mathbf{V}') in terms of the eigenvalues as follows:

$$(\mathbf{V}')_{i,i+t} = \frac{1}{q} \left(\lambda_0^t + 2 \sum_{s=1}^{(q-1)/2} \lambda_s^t \cos \frac{2\pi st}{q} \right) \quad (q \text{ odd}), \quad (38)$$

$$(\mathbf{V}')_{i,i+t} = \frac{1}{q} \left(\lambda_0^t + 2 \sum_{s=1}^{q/2-1} \lambda_s^t \cos \frac{2\pi st}{q} + (-1)^t \lambda_{q/2}^t \right) \quad (q \text{ even}) \quad (39)$$

for $t = 0, 1, 2, \dots$. From these relations we can in principle calculate the partition function of any finite cluster in terms of the eigenvalues.

If we take the particular case of $r = 1$, we rederive the original matrix \mathbf{V} , and hence we have the relations between the interaction function $f(t)$ defined in (33) and the λ_s :

$$f(t) = \frac{1}{q} \left(\lambda_0 + 2 \sum_{s=1}^{(q-1)/2} \lambda_s \cos \frac{2\pi st}{q} \right) \quad (q \text{ odd}) \tag{40}$$

$$f(t) = \frac{1}{q} \left(\lambda_0 + 2 \sum_{s=1}^{q/2-1} \lambda_s \cos \frac{2\pi st}{q} + (-1)^t \lambda_{q/2} \right) \quad (q \text{ even}). \tag{41}$$

We now use the laddering transformation depicted in figure 2, in which the interaction J^* is replaced by J' and J'' in parallel. The function $f(t^*)$ is replaced by $f(t')f(t'')$ (in an obvious notation); using (34) or (35) for λ_t^* in terms of $f(t^*)$ or $f(t')f(t'')$, and then (40) or (41) to transform these back to λ'_t, λ''_t we derive an 'addition theorem' for the eigenvalues under the laddering transformation.

The detailed calculation is somewhat tedious but the result is simple and straightforward. We find that for the lowest eigenvalue

$$q\lambda_0^* = \lambda'_0\lambda''_0 + 2 \sum_{s=1}^{(q-1)/2} \lambda'_s\lambda''_s \quad (q \text{ odd}) \tag{42}$$

$$q\lambda_0^* = \lambda'_0\lambda''_0 + 2 \sum_{s=1}^{q/2-1} \lambda'_s\lambda''_s + \lambda'_{q/2}\lambda''_{q/2} \quad (q \text{ even}). \tag{43}$$

For the higher eigenvalues, writing

$$q\lambda_t^* = \sum d_{ab}^{(t)} \lambda'_a \lambda''_b, \tag{44}$$

it is clear by symmetry that $d_{ab}^{(t)}$ is equal to $d_{ba}^{(t)}$, and we can confine attention to the case $a \geq b$. We then find that $d_{ab}^{(t)}$ is equal to 1 when the following conditions are satisfied, and is otherwise zero:

$$\begin{aligned} a - b &= t \\ a + b &= t \\ a + b &= q - t. \end{aligned} \tag{45}$$

The coefficient q which multiplies the left-hand side of (42), (43) and (44) arises because of the insertion of a new vertex in the laddering procedure.

We quote a few specific examples. When $q = 3$ the transformation is

$$\begin{aligned} 3\lambda_0^* &= \lambda'_0\lambda''_0 + 2\lambda'_1\lambda''_1 \\ 3\lambda_1^* &= \lambda'_0\lambda''_1 + \lambda'_1\lambda''_0 + \lambda'_1\lambda''_1. \end{aligned} \tag{46}$$

Writing $w = \lambda_1/\lambda_0$ we rederive the transformation (23) with $q = 3$, since in this case both Potts models are identical.

When $q = 5$ the transformation is

$$\begin{aligned} 5\lambda_0^* &= \lambda'_0\lambda''_0 + 2\lambda'_1\lambda''_1 + 2\lambda'_2\lambda''_2 \\ 5\lambda_1^* &= \lambda'_0\lambda''_1 + \lambda'_1\lambda''_0 + \lambda'_1\lambda''_2 + \lambda'_2\lambda''_1 + \lambda'_2\lambda''_2 \\ 5\lambda_2^* &= \lambda'_0\lambda''_2 + \lambda'_2\lambda''_0 + \lambda'_1\lambda''_2 + \lambda'_2\lambda''_1 + \lambda'_1\lambda''_1. \end{aligned} \tag{47}$$

The term $\lambda'_2\lambda''_2$ in the second line arises from the third condition in (45); likewise the terms $\lambda'_1\lambda''_2, \lambda'_2\lambda''_1$ in the third line.

When $q = 6$ the transformation is

$$\begin{aligned}
 6\lambda_0^* &= \lambda'_0\lambda''_0 + 2\lambda'_1\lambda''_1 + 2\lambda'_2\lambda''_2 + \lambda'_3\lambda''_3 \\
 6\lambda_1^* &= \lambda'_0\lambda''_1 + \lambda'_1\lambda''_0 + \lambda'_1\lambda''_2 + \lambda'_2\lambda''_1 + \lambda'_2\lambda''_3 + \lambda'_3\lambda''_2 \\
 6\lambda_2^* &= \lambda'_0\lambda''_2 + \lambda'_0\lambda''_3 + \lambda'_1\lambda''_3 + \lambda'_3\lambda''_1 + \lambda'_2\lambda''_2 + \lambda'_1\lambda''_1 \\
 6\lambda_3^* &= \lambda'_0\lambda''_3 + \lambda'_3\lambda''_0 + 2(\lambda'_1\lambda''_2 + \lambda'_2\lambda''_1).
 \end{aligned}
 \tag{48}$$

Again there are a number of terms corresponding to the third condition in (45).

It is interesting to take the transformation to the limit when $q \rightarrow \infty$ when the eigenvalues are given by (36). The third condition in (45) ceases to have any significance, and the transformation corresponds to the well known addition theorem for Bessel functions (Watson 1944, p 365)

$$I_n(K' + K'') = \sum_{m=-\infty}^{\infty} I_{n+m}(K')I_m(K'').
 \tag{49}$$

The partition function of a polygon is given by the trace of \mathbf{V}^q , and is therefore

$$\begin{aligned}
 \lambda_0^r + 2 \sum_{s=1}^{(q-1)/2} \lambda_s^r & \quad (q \text{ odd}) \\
 \lambda_0^r + \lambda_{q/2}^r + 2 \sum_{s=1}^{q/2-1} \lambda_s^r & \quad (q \text{ even}).
 \end{aligned}
 \tag{50}$$

From this we can use the transformations to derive the partition functions of all star topologies which can be derived by laddering. For example, for the θ topology in figure 1 if we write

$$Z(\theta) = q^2 \sum c_{rst} \lambda_r \lambda'_s \lambda''_t,
 \tag{51}$$

where by symmetry $c_{rst} = c_{srt} = c_{trs} \dots$, we find the following values for the nonzero c_{rst} :

$$q = 5: \quad c_{000} = 1, \quad c_{110} = c_{220} = c_{211} = c_{221} = 2
 \tag{52}$$

$$q = 6: \quad c_{000} = c_{330} = 1, \quad c_{110} = c_{220} = c_{211} = c_{321} = c_{222} = 2.
 \tag{53}$$

In the limit as $q \rightarrow \infty$,

$$c_{000} = 1, \quad c_{r+s,r,s} = 2.
 \tag{54}$$

For non-ladder topologies the partition functions are more difficult to calculate for the planar model than for the standard Potts model. Techniques similar to those described for the plane rotator model should be applicable (Domb 1972b), but we defer a detailed discussion to a subsequent publication.

6. Critical temperatures

For the Ising model ($q = 2$) the critical value w_c can be simply related to the geometric properties of self-avoiding walks (SAW) (Domb 1970, 1972a). In fact the SAW limit, μ , provides a good approximation to $1/w_c$ in three dimensions, the error being about 2%.

This is because there are no correction terms of order $1/\mu$, the first correction term being of order $1/\mu^2$.

When we use the same approach for the Potts model with $q > 2$ we find that the above result is no longer valid. Because of the contributions of configurations of odd vertices there is now a first-order correction term which becomes increasingly significant as q becomes larger. Using the self-avoiding N -gon as a basis for classification of terms of order w^N in $\ln Z$ as in Domb (1972a), we can write these terms in the form

$$(q-1)p_N w^N \exp N(B_1 w + B_2 w^2 + \dots), \tag{55}$$

where p_N is the lattice constant of the N -gon, which has asymptotic value

$$p_N \sim \mu^N N^g. \tag{56}$$

B_1 contains contributions from $\theta(N; 1)$ graphs, and B_2 from $\theta(N; 1)$, $\theta(N; 2)$, $\beta(N; 1, 1)$, $\gamma(N; 1, 1)$ and $\alpha(N; 1, 1)$ in the notation of Domb (1972a). In terms of the statistics of contacts on SAW

$$B_1 = (q-2)k(1) \tag{57}$$

$$B_2 = -(q-1)k(1) + (q-2)k(2) + (q-2)^2 k(1, 1) + (q-1)(q-2)(q-3)k(1, 1; \alpha).$$

The critical temperature to second order in $1/\mu$ is then given by

$$w_c = \frac{1}{\mu} \left(1 - \frac{B_1}{\mu} - \frac{B_2 - \frac{3}{2}B_1^2}{\mu^2} \right). \tag{58}$$

Substituting numerical values for the face-centred cubic (FCC) lattice we find that

$$\begin{aligned} B_1 &\simeq 0.72(q-2) \\ B_2 &\simeq -0.72(q-1) + 5.8(q-2) - 0.4(q-2)^2 + 0.7(q-1)(q-2)(q-3). \end{aligned} \tag{59}$$

For the plane triangular lattice in two dimensions the estimates of the contact statistics are much rougher,

$$\begin{aligned} B_1 &\simeq 0.59(q-2) \\ B_2 &\simeq -0.59(q-1) + 2.5(q-2) - 0.2(q-2)^2 + 0.3(q-1)(q-2)(q-3). \end{aligned} \tag{60}$$

We note that the deviations from the SAW approximation are opposite in sign when $q \geq 3$ from the Ising model ($q = 2$), and they increase steadily with increasing q . The qualitative behaviour as a function of q follows that of the quadratic lattice for which w_c is known exactly,

$$w_c = \frac{\sqrt{q}-1}{q-1}. \tag{61}$$

Taking $q = 3$ we obtain the following numerical estimate for the FCC lattice ($\mu = 10.035$):

$$\frac{1}{w_c} \simeq 11.1(5). \tag{62}$$

The corresponding Ising estimate to this order is 9.96, the correct value being 9.828.

The planar Potts model with $q = 3$ is identical with the standard Potts model above. However, once $q > 3$ the pattern changes completely and all first-order terms in $1/\mu$ are zero. When $q = 4$ the critical temperature is the same as with $q = 2$ (Ising model).

When $q \geq 5$ the value of w_c to second order in $1/\mu$ is the same as for $q = \infty$,

$$w_c = \frac{1}{\mu} \left(1 - \frac{1.5k(1)}{\mu^2} \right), \quad (63)$$

and only higher-order terms are affected†. We thus see that w_c for the planar Potts model converges rapidly with increasing q to the value of w_c for the plane-rotator model.

7. Conclusions

Our major aim in the present paper has been to lay the foundation for the development of extensive high-temperature series expansions for the Potts models analogous to those which have been developed for the Ising model. We have found general indications that such series should converge reasonably in three dimensions, but that the rate of convergence will be slower than for the Ising model, particularly as q increases.

We have not discussed the susceptibility which diverges strongly and is important for obtaining an accurate estimate of T_c . This susceptibility can be defined in different ways (Straley and Fisher 1973). From the point of view of the discussion in the present paper the most useful quantity to calculate is the sum of the pair correlations, which can be simply related to the partition functions of finite clusters.

We hope to discuss the detailed calculations in a subsequent publication.

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† The value given in Domb (1970) equation (84) is in error by a factor of 2 in the $1/\mu^2$ term. The corresponding formula for CH is in error by a factor of 3 and should read $-1.8/\mu^2$.

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